THE GEOMETRIC BASIS
of
ANALYTIC TRIANGULATION

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NOTICES

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FOREWORD

Recent ACIC activities in the field of analytic photogrammetry have developed a requirement among technicians for knowledge of the mathematical essentials of analytic aerotriangulation. This manual is one of a series of manuals designed to provide the mathematical knowledge required to understand current literature related to ACIC activities (other manuals already printed include Technical Memorandum TM-17, "Determinants and Matrices" and TM-16 "Functional Notation," prepared in the Research Division). The material presented reoresents information relative to the geometric basis for solution to the aerotriangulation problem, which is fundamentally the projective relations between a point, a line, and a plane in space.

There are six chapters in this manual. The first chapter gives a geometric definition of the general aerotriangulation problem and reviews some fundamental propositions from solid geometry which are necessary for further study. Subsequent chapters deal with the point in space, the line in space, the plane in space, and systems of coordinates with matrix notation for transforming coordinates from one reference system to another system. The transformation from a space coordinate system to a geodetic coordinate system based on a spheroidal earth is a specific consideration. The last chapter deals with practical applications.

GEOMETRIC BASIS OF ANALYTIC AEROTRIANGULATION

1. INTRODUCTION

1.1 The General Problem.

The general purpose of analytic aerotriangulation is to establish ground positions from the rectangular coordinates of objects recorded on photographs or photographic plates. The aerotriangulation problem is basically the same as any other measuring problem -- precise measurement and the exact mathematical expression of the geometric relationships in the problem.

The following sections are introductory to any treatment of the analytic aerotriangulation problem. The entire analytical solution is based on a few theorems and concepts from plane and solid analytical geometry. This manual is designed to assist the reader in understanding the use of these concepts in the aerotriangulation problem.

The presentation emphasizes the projective relationships between a point and plane in space, a line and plane in space, and planes in space. Even though these topics are interrelated and could very well be included in one chapter, each topic (the point, the line and the plane in space) is treated separately to show its individual application to the problem. Only those derivations necessary to visualize the geometric concepts are discussed, including the transformation of coordinates in space and systems of coordinates.

1.2 Fundamental Definitions and Propositions. The following basic definitions and propositions from solid geometry are essential to an understanding of the material in the in the next four chapters:

a. The locus of a point is the path of the point when it moves according to some definite law. In plane geometry, a locus is usually a straight line or a curve. In solid geometry, a locus is usually a plane or a curved surface.

b. A straight line is the locus of a point in a plane always moving in a fixed direction. Generally, a surface is the locus of a point in space which satisfies one given condition. The locus of a point in space which satisfies two conditions is generally a curve.

c. A ray is a line extending an infinite distance in one direction from a fixed point.

d. A segment is a limited portion of a line.

e. A plane is determined by:

(1) A straight line and a point outside the line.

(2) Three points which are not on the same straight line.

(3) Two intersecting lines or two parallel lines.

f. The orthogonal projection of a point on a plane is the foot of a perpendicular drawn from the point to the plane.

g. The projection of a line on a plane is the line containing the projections of all points of the given line on the given plane.

h. Two planes are parallel if they never meet, however far they are extended.

i. One and only one plane parallel to the given plane can be drawn through a point outside a plane.

j. All the perpendiculars to a given line at a point in the line lie in a plane perpendicular to the line at that point.

k. Two planes are perpendicular if they form a right dihedral angle.

l. If a line is perpendicular to a plane, every plane containing the line is perpendicular to the given plane.

m. Through a given line not perpendicular to a given plane, one plane and only one plane can be drawn perpendicular to the given plane.

2. THE POINT IN SPACE

2.1 Rectangular Coordinates in Space. The idea of rectangular coordinates as developed in plane analytical geometry may be extended to space by assuming three mutually perpendicular planes intersecting in the lines XX', YY', and ZZ', Figure 1.

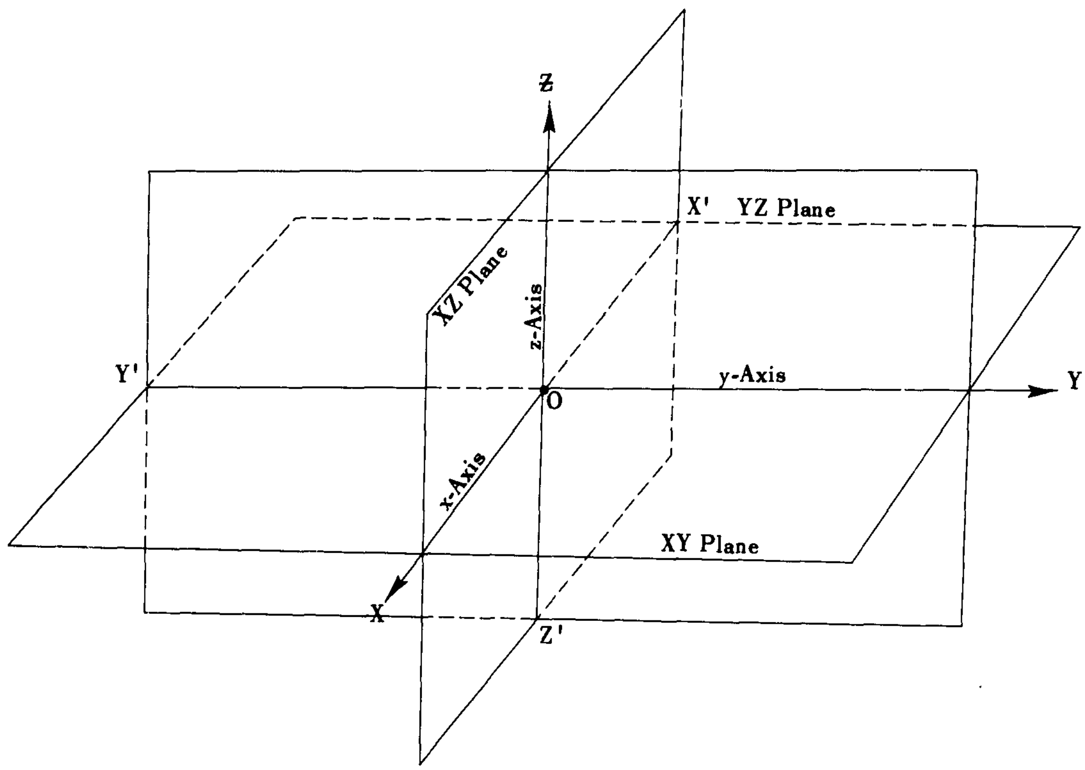

Figure 1

The planes XY, YZ and XZ are coordinate planes and their lines of intersection establish the y-axis, z-axis and x-axis. The positive directions are indicated by the arrowheads and the point of intersection is the origin. The eight sections of space separated by the coordinate planes are called octants. Let P in Figure 2 be in the first octant: the octant behind the first is numbered the second, the octant to the left of the second is the third, and the octant in front of the third (and to the left of the first) is the fourth. By starting with the octant directly below the first and proceeding in a similar manner, the fifth, sixth, seventh and eighth octants, respectively, are obtained.

The octant in which a point is located can be determined by the signs or direction of its coordinate axes. The following table illustrates the classification of octants according to the signs of the coordinate axes:

Octant	x-axis	y-axis	z-axis
1st	+	+	+
2nd	–	+	+
3rd	–	–	+
4th	+	–	+
5th	+	+	–
6th	–	+	–
7th	–	–	–
8th	+	–	–

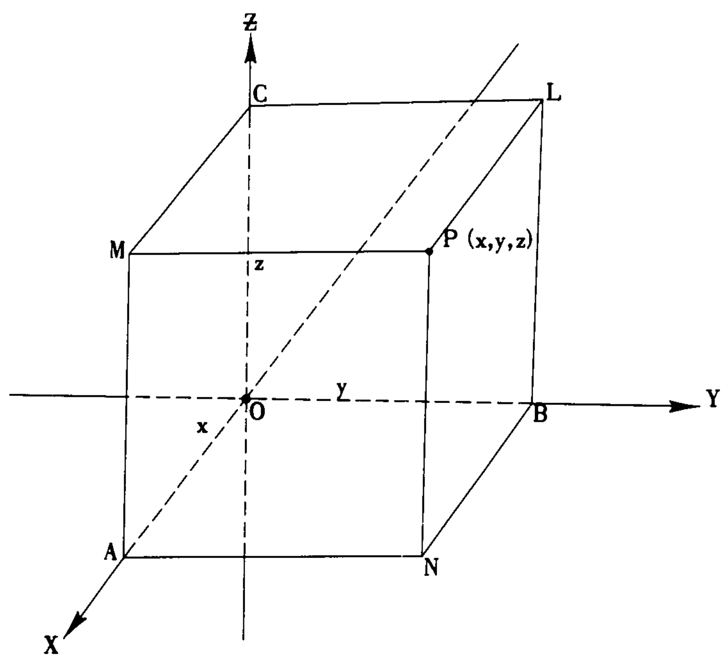

Figure 2

In Figure 2, three planes are drawn through the point P parallel to the three coordinate planes and cutting the axes at A, B and C. The distances, OA = x, OB = y, and OC = z are the rectangular coordinates of P. They are equal numerically to the perpendicular distances from the coordinate planes and designated (x, y, z) or for a point in the fourth octant (2, –3, 4).

2.2 Equations of a Point in Space. Since three planes through the point P in Figure 2 are drawn parallel to the coordinate planes, which are mutually perpendicular, then OA = LP, OB = MP and OC = NP. The lines LP, MP and NP are perpendicular to the ZY, XZ and XY planes respectively. By letting LP = a, MP = b and NP = c, the three equations

$$x = a, \ y = b, \ z = c \tag{2-1}$$

are the equations of the point P. For the point N, in the XY plane, z = 0 and the equations of N are

$$x = a, \ y = b, \ z = 0 \tag{2-2}$$

For the point A, on the x-axis, both y and z are zero, and the equations of A are

$$x = a, \ y = 0, \ z = 0 \tag{2-3}$$

In every case, three equations are necessary to determine a point in three-dimensional space.

2.3 Distance Between Two Points in Space.

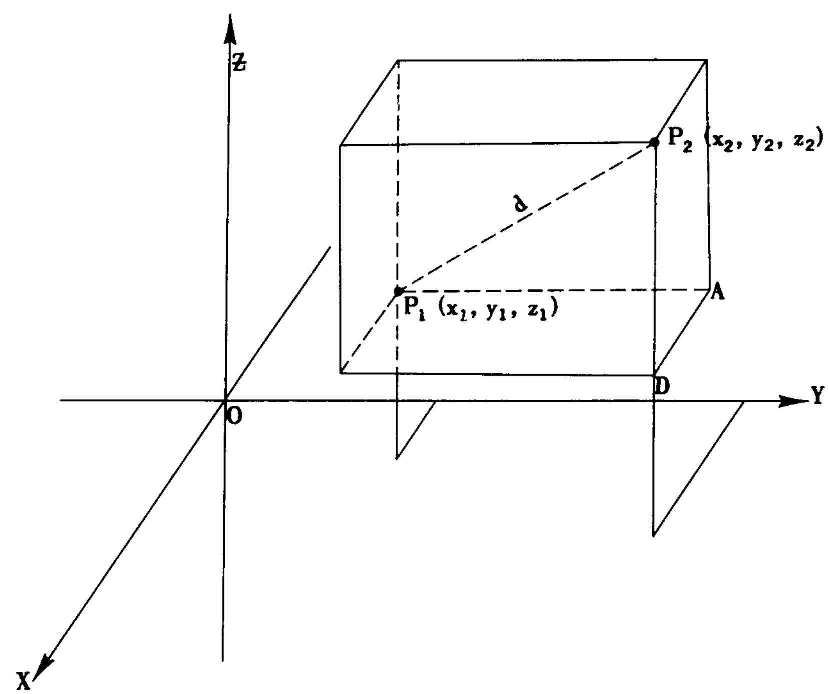

Figure 3

Let P_1 and P_2 (Figure 3) be two points in space with coordinates x_1, y_1, z_1 and x_2, y_2, z_2 respectively. By applying the Pythagorean theorem

$$(P_1D)^2 = (P_1A)^2 + (AD)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 \tag{2-4}$$

and

$$(P_1P_2)^2 = (P_1D)^2 + (DP_2)^2 \tag{2-5}$$

where $(DP_2)^2 = (z_2 - z_1)^2$. By substituting the values of $(P_1D)^2$ and $(DP_2)^2$ in equation (2-5) and denoting $(P_1P_2)^2$ by d^2, the distance between two points in space is determined by the formula

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$$

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \tag{2-6}$$

If P_1 is the origin (0,0,0) and the coordinates of P_2 are x,y,z, then

$$d = \sqrt{(x - 0)^2 + (y - 0)^2 + (z - 0)^2} \text{ or}$$

$$d = \sqrt{x^2 + y^2 + z^2} \tag{2-7}$$

2.4 Orthogonal Projection of a Point in Space. The orthogonal projection of a point on a coordinate plane is the foot of the perpendicular from the point to the plane.

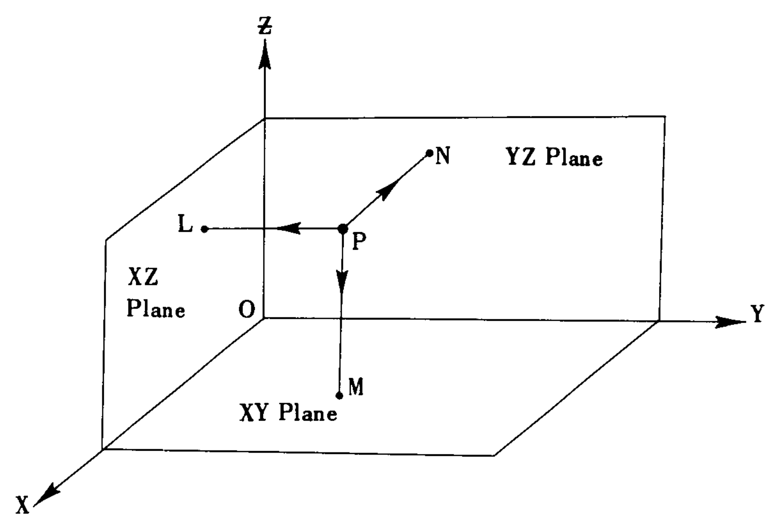

Figure 4

In Figure 4, L is the orthogonal projection of the point P upon the XZ plane, M is the orthogonal projection of P upon the XY plane, and N is the orthogonal projection of P upon the YZ plane.

3. THE LINE IN SPACE.

3.1 Projections of a Line Upon the Coordinate Planes. Two non-parallel planes in in space will intersect in one and only one straight line. The projection of a line $P_1 P_2$, upon any one of the three coordinate planes is the line of intersection of the coordinate plane with a plane containing the line $P_1 P_2$. The projection is the orthogonal projection of the line $P_1 P_2$ if the projecting plane is perpendicular to the coordinate plane (Figure 5).If the projecting plane is not perpendicular, the projection is oblique.

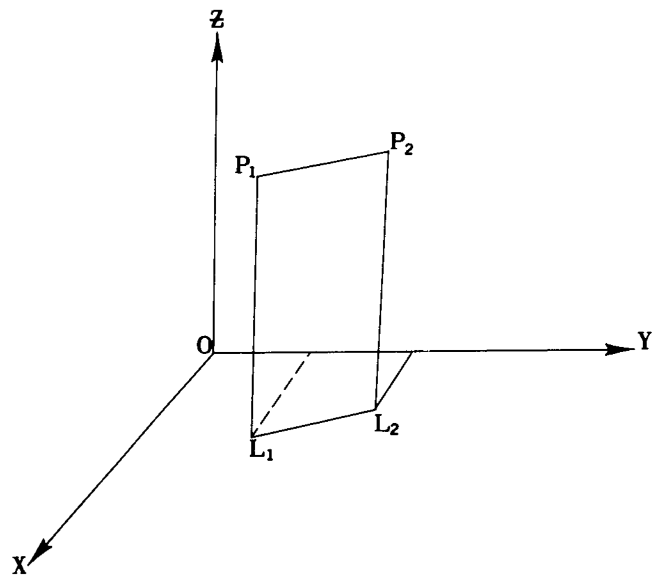

Figure 5

3.2 Direction Numbers and Direction Cosines of a Line. Even though two lines in space do not intersect, it is convenient to define the angle between them as the angle between intersecting lines which are respectively parallel to them. Let O'P'Figure 6 be any directed line in space, and OP be a line through the origin O which has the same direction and is parallel to O'P'. The angles α, β, and γ are angles which OP makes with the coordinate axes. These are direction angles and their cosines are direction cosines

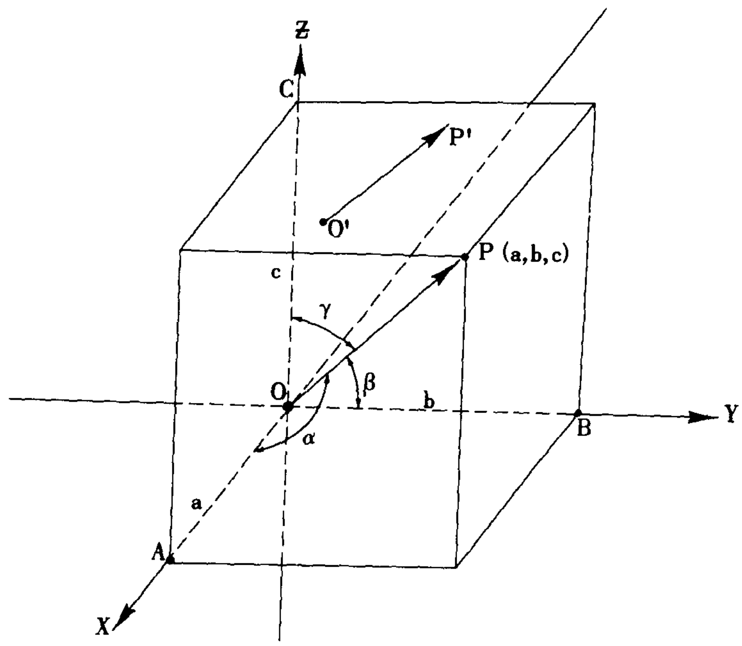

Figure 6

of the line OP. The latter will be denoted by λ, μ, and ν respectively. Let P be denoted by the coordinates (a, b, c), then in the triangle OPA

$$\lambda = \cos\alpha = \frac{a}{OP} \tag{3-1}$$

Similarly in triangles OBP and OCP respectively,

$$\mu = \cos\beta = \frac{b}{OP}, \quad \nu = \cos\gamma = \frac{c}{OP}. \tag{3-2}$$

Since OP is the diagonal of a parallelepiped whose edges are a, b and c,

$$OP = \sqrt{a^2 + b^2 + c^2}. \tag{3-3}$$

By substituting the value of OP in Equations (3-1) and (3-2), the direction cosines can be expressed as

$$\lambda = \cos\alpha = \frac{a}{\pm\sqrt{a^2 + b^2 + c^2}}$$

$$\mu = \cos\beta = \frac{b}{\pm\sqrt{a^2 + b^2 + c^2}} \qquad (3\text{-}4)$$

$$\nu = \cos\gamma = \frac{c}{\pm\sqrt{a^2 + b^2 + c^2}}$$

By squaring each member of the latter equations and adding the results, the fundamental formula of space trigonometry is obtained:

$$\cos^2\alpha + \cos\hat{}\,\beta + \cos\gamma = 1. \text{ or } \lambda^2 + \mu^2 + \nu^2 = 1 \qquad (3\text{-}5)$$

The sum of the squares of the direction cosines of a line is equal to unity.

Any three numbers proportional to the direction cosines of a line are the <u>direction numbers</u> of the line. By definition

$$\frac{\cos\alpha}{a} = \frac{\cos\beta}{b} = \frac{\cos\gamma}{c} = r$$

If r represents the common value of the ratios, then,

$$\cos\alpha = ar, \cos\beta = br, \cos\gamma = cr. \qquad (3\text{-}6)$$

Squaring, adding, and applying Equation (3-5)

$$r^2(a^2 + b^2 + c^2) = 1, \text{ and}$$

$$r = \frac{1}{\pm\sqrt{a^2 + b^2 + c^2}}.$$

Substituting this value for r in Equation (3-6), formulas for direction cosines (3-4) are obtained. (The importance of this derivation is that any three numbers a, b, and c determine the direction of a line in space. This direction is the same as that of the line joining the origin and the point (a, b, c) when the positive sign of the radical is chosen, otherwise the direction is reversed.)

If one direction number is zero, the line is perpendicular to the corresponding axis of the coordinates. If two are zero, it is parallel to the remaining axis. The following considerations will guide the determination of signs for the radical in Equation (3-4):

a. If a line cuts the XY-plane, it will be directed upward or downward according as $\cos\gamma$ is positive or negative.

b. If a line is parallel to the XY-plane, $\cos\gamma = 0$, and the line will be directed in front or in the back of the ZX-plane according as $\cos\alpha$ is positive or negative.

c. If a line is parallel to the x-axis, $\cos\beta = \cos\gamma = 0$, and its positive direction will agree or disagree with that of the x-axis according as $\cos\alpha = 1$ or -1.

3.3 Equation of a Line in Space. In plane analytic geometry, the equation of a straight line on the XY plane is:

$$Ax + By + C = 0, \tag{3-7}$$

where A, B, and C are constants.

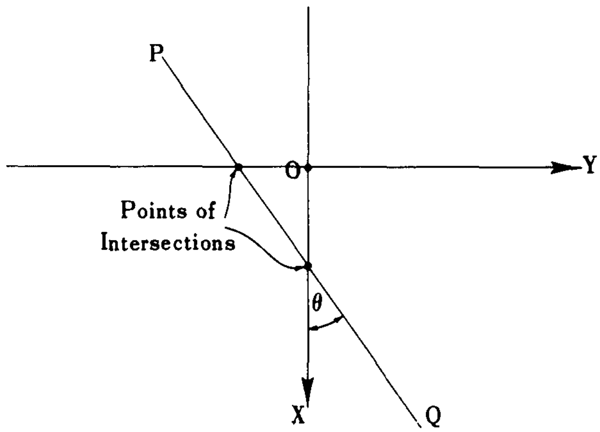

Figure 7

Let the line PQ intersect the y-axis at the point $y = -\frac{C}{B}$, and makes an angle θ with the x-axis such that

$$\tan\theta = -\frac{A}{B}, \text{ Figure 7.} \tag{3-8}$$

Consider this same equation in three-dimensional space. The same relationship between x and y holds with no limit on the z-coordinate. The locus of the equation is a plane perpendicular to the XY coordinate plane and makes the angle θ with the XZ coordinate plane (Equation 3-8, and Figure 8).

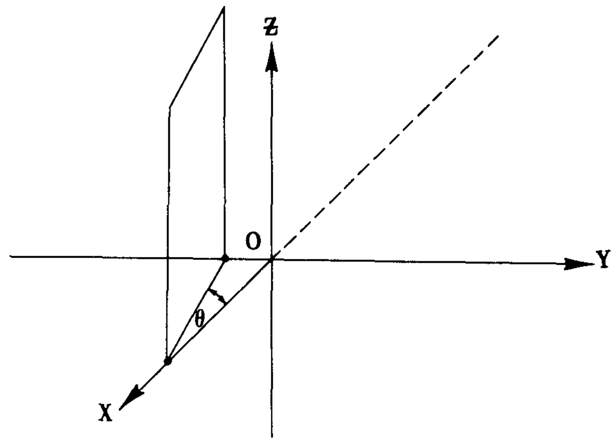

Figure 8

The theorem from solid geometry which states that two non-parallel planes intersect in a straight line leads to the conclusion that one form of the equations of a straight line can be developed from any two of the three projections of the line on the coordinate planes. Thus any two of the following three equations are the equations of a line in space:

$$A_1x + B_1y + C_1 = 0$$
$$A_2x + B_2z + C_2 = 0 \qquad (3\text{-}9)$$
$$A_3y + B_3z + C_3 = 0$$

where A, B, and C are constants.

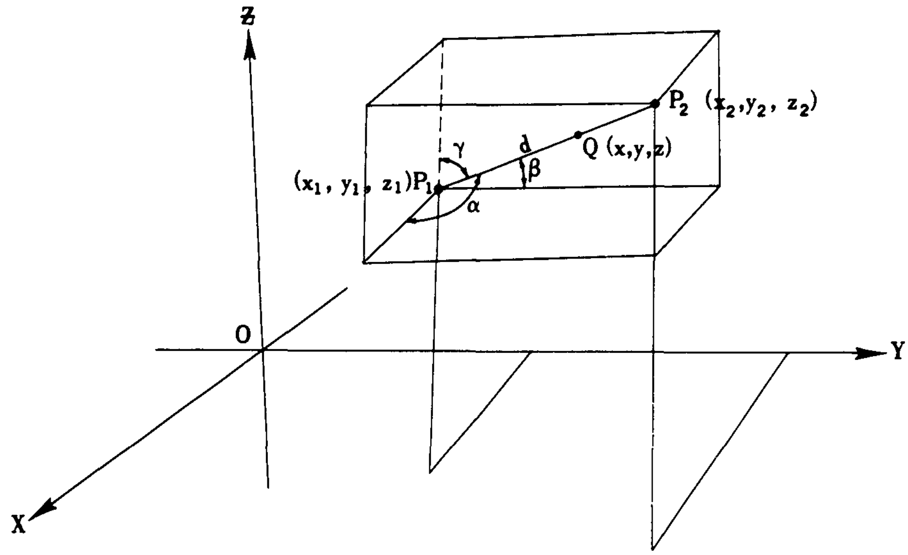

Figure 9

In Figure 9, let P_1 and P_2 be two known points on a line of known length d with known direction cosines α, β, and γ. Let Q be a variable point with coordinates x, y, z. Then

$$\frac{x - x_1}{d} = \cos\alpha\,,$$

$$\frac{y - y_1}{d} = \cos\beta\,, \tag{3-10}$$

$$\frac{z - z_1}{d} = \cos\gamma\,,$$

and by simple transformation

$$x = x_1 + d\cos\alpha$$

$$y = y_1 + d\cos\beta \tag{3-11}$$

$$z = z_1 + d\cos\gamma$$

The equations in (3-11) are the parametric form of the equations of a line in space. solving Equations (3-11), for d and equating the results:

$$\frac{x - x_1}{\cos\alpha} = \frac{y - y_1}{\cos\beta} = \frac{z - z_1}{\cos\gamma} \tag{3-12}$$

which yields the symmetric form of the equations of a line in space. If a, b, and c are

direction numbers the symmetric form may be written in the form

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}, \tag{3-13}$$

since direction numbers are proportional to the direction cosines.

The equation of the straight line passing through two known points P_1 and P_2 is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} \tag{3-14}$$

Since the denominators in Equation (3-14) are direction numbers, the two-point form follows directly from the symetric form.

The simultaneous equations

$$x = mz + p, \; y = nz + q \tag{3-15}$$

define a line as the intersections of its projecting planes on XZ and YZ. This is the projection form of the equations of a line. Separately, each equation is the equation of a projecting plane. The equations are identical to the point slope form of the equation of a line on a plane, where m and n are slopes; i.e., $\tan \theta_1 = m$, $\tan \theta_2 = n$, and p and q are intercepts. Solving for z and equating results

$$\frac{x - p}{m} = \frac{y - q}{n} = \frac{z}{1}. \tag{3-16}$$

Comparison with the symmetric form shows that in the projection form the line passes through (p, q, o) and m, n and 1 are direction numbers. Since m, n, and 1 are direction numbers and proportional to the direction cosines, the relationship of the direction cosines to the slopes of the projecting planes can be derived directly from the equations. Let k be the constant of proportionality and,

$$mk = \cos \alpha, \; nk = \cos \beta, \; k = \cos \gamma \tag{3-17}$$

Squaring and adding

$$\begin{aligned} m^2k^2 + n^2k^2 + k^2 &= k^2(m^2 + n^2 + 1) \\ &= \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1 \end{aligned} \tag{3-18}$$

Solving for k

$$\begin{aligned} k^2 &= \frac{1}{m^2 + n^2 + 1} \\ k &= \frac{1}{\sqrt{m^2 + n^2 + 1}} \end{aligned} \tag{3-19}$$

Substituting this value of k in Equation (3-17)

$$\cos\alpha = \frac{m}{\sqrt{m^2 + n^2 + 1}}$$

$$\cos\beta = \frac{n}{\sqrt{m^2 + n^2 + 1}} \tag{3-20}$$

$$\cos\gamma = \frac{1}{\sqrt{m^2 + n^2 + 1}}$$

3.4 Condition for Intersection of Two Lines in Space.

Let

$$x = mz + p \tag{3-21}$$

$$y = nz + q \tag{3-22}$$

and

$$x = m'z + p' \tag{3-23}$$

$$y = n'z + q' \tag{3-24}$$

be the equations of any two lines in space. In general, these lines will not intersect, since the coefficient m, n, p and q are entirely arbitrary. The four equations contain three unknowns, x, y, z which are not always consistent, even though the lines in space do not intersect, the respective projection will, unless they are parallel. By eliminating x in Equations (3-21) and (3-23), and y in Equations (3-22) and (3-24), then,

$$z = \frac{p' - p}{m - m'} \tag{3-25}$$

and

$$z = \frac{q' - q}{n - n'} \tag{3-26}$$

Equations (3-25) and (3-26) give the value of the z-coordinate of the point of intersection of the xz- and yz- projections of the two lines. These values usually are unequal; however if the lines intersect, the values of z will be equal. Conversely, if the values are equal, the lines intersect. The necessary and sufficient condition that two lines in space shall intersect is determined by equating (3-25) and (3-26), such that

$$\frac{p' - p}{m - m'} = \frac{q' - q}{n - n'} \tag{3-27}$$

Equation (3-27) gives the condition for intersection and, at the same time, the value of the z-coordinate of the point of intersection. To find the other coordinates, substitute the value of z from (3-25) in (3-21) or (3-23) and from (3-26) in (3-22) or (3-24), the resulting values will be,

$$x = \frac{mp' - m'p}{m - m'} \tag{3-28}$$

$$y = \frac{nq' - n'q}{n - n'} \tag{3-29}$$

As a numerical example of the material presented in this section, let

$$x = 3z - 2 \qquad\qquad x = -7z + 8$$
$$\text{and}$$
$$y = -4z + 5 \qquad\qquad y = z$$

then,

$$m = 3, \quad p = -2, \quad n = -4, \quad q = 5$$
$$m' = -7, \quad p' = 8, \quad n' = 1, \quad q' = 0$$

Substituting these values in (3-27),

$$\frac{8 + 2}{3 + 7} = \frac{0 - 5}{-4 - 1}$$

or $\qquad 1 = 1$

The example shows that the lines do intersect at the point where the z-coordinate is 1. By substituting the value of 1 in either set of xz- and yz projection equations, $x = 1$ and $y = 1$ and the two lines intersect at the point (1, 1, 1).

3.5 The Angle Between Two Lines.

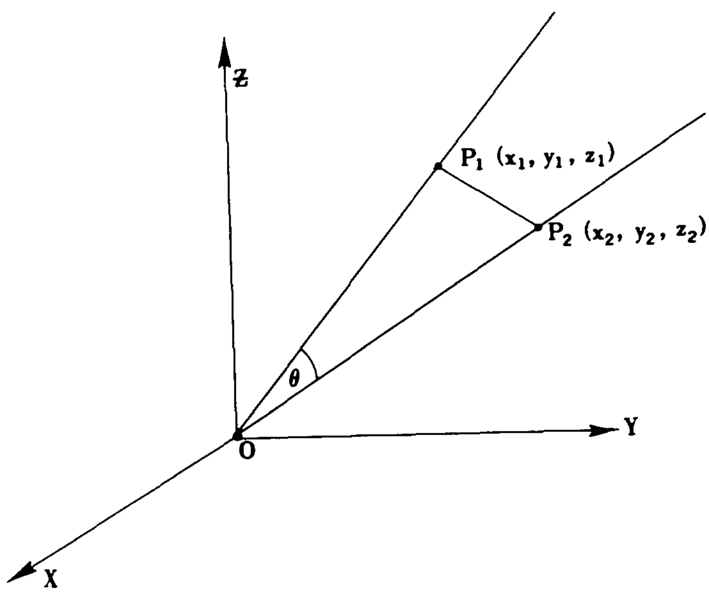

Figure 10

Let OP_1 and OP_2 (see Figure 10) be two directed lines in space, such that

$$(OP_1)^2 = x_1^2 + y_1^2 + z_1^2$$

and $$(OP_2)^2 = x_2^2 + y_2^2 + z_2^2; \tag{3-30}$$

also $$(P_1P_2)^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2.$$

Applying the Law of Cosines from Plane Trigonometry,

$$(P_1P_2)^2 = (OP_1)^2 + (OP_2)^2 - 2\,(OP_1)\,(OP_2)\cos P_1\,O\,P_2 \tag{3-31}$$

where $$\cos P_1\,O\,P_2 = \frac{(OP_1)^2 + (OP_2)^2 - (P_1P_2)^2}{2\,(OP_1)\,(OP_2)}. \tag{3-32}$$

By substituting in (3-31) the values obtained in (3-30) and letting angle $P_1\,OP_2$ be represented by θ, then:

$$\cos\theta = \frac{x_1x_2 + y_1y_2 + z_1z_2}{(OP_1)\,(OP_2)}. \tag{3-33}$$

From the definition discussed with direction cosines:

$$x_1 = OP_1\cos\alpha_1,\ y_1 = OP_1\cos\beta_1,\ z_1 = OP_1\cos\gamma_1$$

and $$x_2 = OP_2\cos\alpha_2,\ y_2 = OP_2\cos\beta_2,\ z_2 = OP_2\cos\gamma_2.$$

Substituting these values in (3-33), the following formula is obtained for the angle between two lines in terms of the direction cosines of the lines:

$$\cos\theta = \cos\alpha_1\cos\alpha_2 + \cos\beta_1\cos\beta_2 + \cos\gamma_1\cos\gamma_2$$

or $$\cos\theta = \lambda_1\lambda_2 + \mu_1\mu_2 + \nu_1\nu_2$$

For parallel lines:

$$\cos\alpha_1 = \cos\alpha_2,\ \cos\beta_1 = \cos\beta_2,\ \cos\gamma_1 = \cos\gamma_2$$

or $$\lambda_1 = \lambda_2,\ \mu_1 = \mu_2,\ \nu_1 = \nu_2$$

For perpendicular lines, $\theta = 90°$ and $\cos\theta = 0$:

$$\cos\alpha_1\cos\alpha_2 + \cos\beta_1\cos\beta_2 + \cos\gamma_1\cos\gamma_2 = 0.$$

or $$\lambda_1\lambda_2 + \mu_1\mu_2 + \nu_1\nu_2 = 0.$$

4. THE PLANE IN SPACE

4.1 Definitions. A plane is defined as a surface such that a line connecting any two points of it will lie wholly within the surface. A plane may be defined as the surface generated by a line rotating about another line to which it is perpendicular. The rotating line is called the generator and the fixed line, or axis, the director. A plane may be determined by three points (not on the same line), or by a line and an external point, or by two intersecting or parallel lines.

4.2 Equations of a Plane.

4.2.1 Normal Form. The normal form of the equation of the plane is

$$x \cos \alpha + y \cos \beta + z \cos \gamma = \rho$$

This important form gives the relationship between the coordinates of all points in a plane in terms of the perpendicular distance from the origin to the plane, and the direction cosines of this perpendicular. Figures 11a and 11b.

In the diagram (11a) the plane LMN represents the position of the plane in one of the octants. The line OD is perpendicular to the plane and intersects the plane at D. The length of the normal OD is denoted by ρ which is considered a positive quantity. The angles α, β and γ are the direction angles of the plane director, ρ. When the value of ρ and the direction cosines are known, the point D can be located, and the plane is

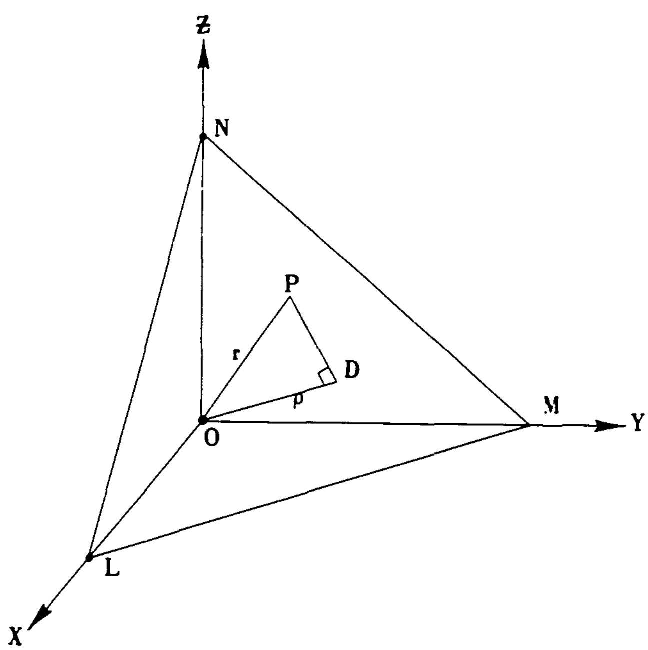

Figure 11a

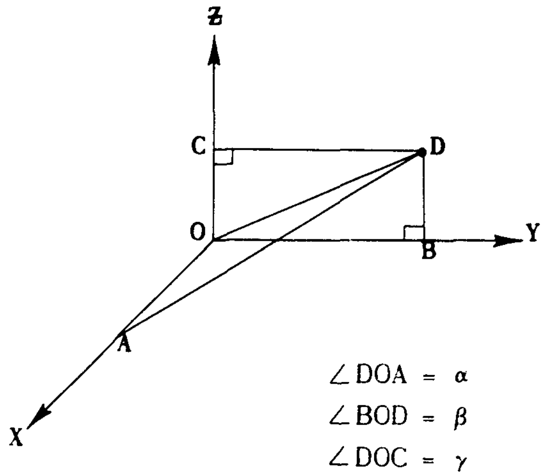

Figure 11b

constructed as a plane through D and perpendicular to OD. P is any point in the plane LMN, whose coordinates are x, y and z. The triangle POD is always a right triangle regardless of what point in the plane is chosen as P.

. If one of the direction angles is a right angle, ρ is perpendicular to one of the coordinate axes and the plane is parallel to that axis. For example, if $\gamma = 90°$ the equation becomes

$$x \cos \alpha + y \cos \beta = \rho$$

(the equation of a plane parallel to the Z-axis). If two of the direction angles are right angles the equation reduces to the form

$$x \cos \alpha = \rho \text{ , } y \cos \beta = \rho \text{ , or } z \cos \gamma = \rho \text{ .}$$

If β and γ are right angles, ρ coincides with the x-axis and the plane is perpendicular to the x-axis.

4.2.2 General Form. Every equation of the first degree can be written in the form:

$$Ax + By + Cz + D = 0$$

and any equation of this form is which A, B, C are not all zero represents a plane. This general equation of a plane is reduced to the normal form by the following rules:

Rule 1. Change all signs so that the absolute term of the given equation is negative in the first member. (ρ is always considered positive).

Rule 2. Divide by the square root of the sum of the squares of the coefficients of coefficients of x, y, z.

The normal form of the general equation is

$$\mp\left(\frac{A_x + B_y + C_z + D}{\sqrt{A^2 + B^2 + C^2}}\right) = 0$$

The value of the direction cosines of the plane director are:

$$\cos\alpha = \mp \frac{A}{\sqrt{A^2 + B^2 + C^2}}, \quad \cos\beta = \mp \frac{B}{\sqrt{A^2 + B^2 + C^2}} \text{ and}$$

$$\cos\gamma = \mp \frac{C}{\sqrt{A^2 + B^2 + C^2}}. \text{ The value of } \rho = \pm \frac{D}{\sqrt{A^2 + B^2 + C^2}}$$

The upper sign is to be used when the absolute term of the given general equation is positive and the lower sign when it is negative.

Several special cases of the general equation are listed below:

Equation	Conditions
$A_x + B_y + D = 0$	$\perp$ xy plane $\parallel$ z-axis
$A_x + C_z + D = 0$	$\perp$ xz plane $\parallel$ y-axis
$B_y + C_z + D = 0$	$\perp$ yz plane $\parallel$ x-axis
$A_x + D = 0$	$\perp$ x-axis $\parallel$ yz plane
$B_y + D = 0$	$\perp$ y-axis $\parallel$ xz plane
$C_z + D = 0$	$\perp$ z-axis $\parallel$ xy plane

4.2.3 Intercepts and Traces. The intercepts of the plane on the respective axes are the distances from the origin, O, to the point of intersection L, M, and N of the plane with the coordinate axes x, y, z. The traces of the given plane are the lines in which a plane intersects the coordinate planes. The lines LM, MN, and NL (Figure 11a) are the traces of the plane on the coordinate planes XY, YZ, and XZ.

The symmetric (intercept) form of the general equation is

$$\frac{x}{-\frac{D}{A}} + \frac{y}{-\frac{D}{B}} + \frac{z}{-\frac{D}{C}} = 1$$

The intercepts on the x, y, z axes are $-\frac{D}{A}$, $-\frac{D}{B}$ and $-\frac{D}{C}$ respectively.

The slope form of the general equation is:

$$\frac{A}{C}x + \frac{B}{C}y + z + \frac{D}{C} = 0$$

Hence, the general equation of the first degree represents a plane where the slopes of the xz and yz projections of its plane director are A/C and B/C respectively and the intercept on the z-axis is equal to $-\frac{D}{C}$. For example, in the plane $2x - 4y - 3z + 12 = 0$ to obtain the symmetric form, transpose the absolute term to the second member and divide by the transposed absolute term to obtain $\frac{x}{-6} + \frac{y}{3} + \frac{z}{4} = 1$. Hence, the intercepts are $x = -6$, $y = 3$, $z = 4$. The plane is shown in Figure 12.

In the same example, to obtain the slope form, divide the given equation by the coefficient of z and obtain:

$$-\frac{2}{3}x + \frac{4}{3}y + z - 4 = 0$$

The slopes of the xy and yz projections of its plane director are $-2/3$ and $4/3$ respectively and the intercept on the z-axis is 4.

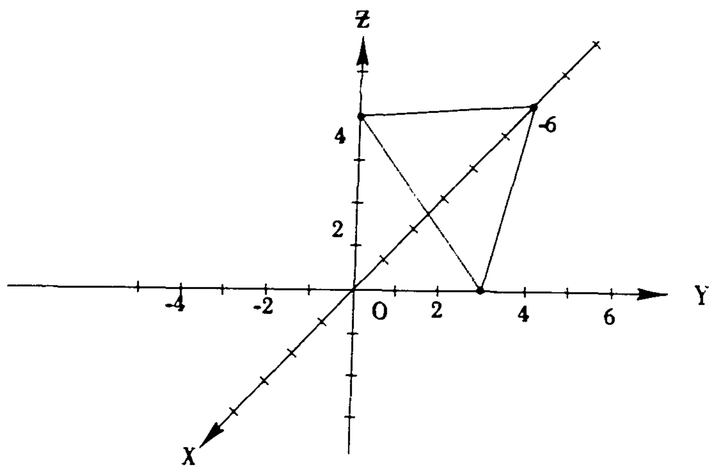

Figure 12

The plane $4x + 5y - 20 = 0$ (Figure 13) has the intercepts $x = 5$ and $y = 4$ but no intercept on the z-axis because it is parallel to that axis. The traces are $4x + 5y - 20 = 0$, $x = 5$, $y = 4$.

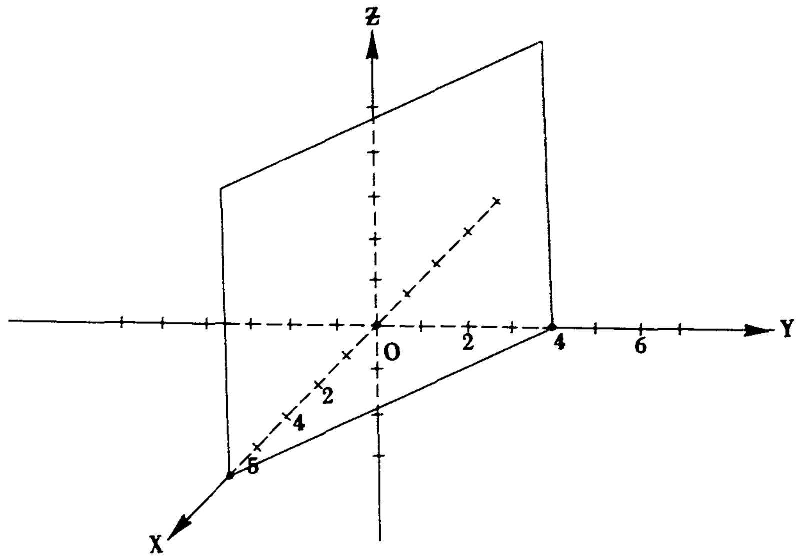

Figure 13

The symmetric or intercept form of the equation of a plane is the most useful form to plot the position of the plane. The values of the intercept are established and plotted on the respective axes. Connecting the ends of the intercepts gives the traces of the plane. In the plane $2x - 4y - 3z + 12 = 0$ the trace in the XY plane ($z = 0$) is $2x - 4y + 12 = 0$. Similarly, the trace in the XZ plane is $2x - 3y + 12 = 0$ and the trace in the YZ plane is $4y + 3z - 12 = 0$. The slope form of the normal equation is

$$\frac{\cos\alpha}{\cos\gamma}x + \frac{\cos\beta}{\cos\gamma}y + z = \frac{\rho}{\cos\gamma}$$

where the coefficients of x and y are the slopes of the plane director.

To change the general Equation $2x - 4y - 3z + 12 = 0$ to the normal form, apply Rule 1 and change all signs because the sign of the absolute term in the left member is positive and obtain $-2x + 4y + 3z - 12 = 0$. To apply Rule 2, divide by $\sqrt{(-2)^2 + 4^2 + 3^2}$ or $\sqrt{29}$ and get the desired normal form,

$$-\frac{2}{\sqrt{29}}x + \frac{4}{\sqrt{29}}y + \frac{3}{\sqrt{29}}z - \frac{12}{\sqrt{29}} = 0$$

so that

$$\cos\alpha = -\frac{2}{\sqrt{29}} \quad \cos\beta = \frac{4}{\sqrt{29}} \quad \cos\gamma = \frac{3}{\sqrt{29}} \text{ and } \rho = \frac{12}{\sqrt{29}}$$

4.3 Distance Between a Point and a Plane. The distance from any given plane to any point in space is found by reducing the equation of the given plane to the normal form, and substituting in the normal form the coordinates of the given point for x, y and z. The resulting numerical quantity expresses the required distance (Figure 14). To find the distance of the point (–3, 4, –5) from the plane $x + 2y - 3z + 8 = 0$ reduce the equation of the plane to normal form (Section 4.2.2.) and obtain

$$\frac{-x - 2y + 3z - 8}{\sqrt{14}} = 0$$

Substituting the coordinates of the given point x, y and z in the normal form obtain

$$\frac{+3 - 8 - 15 - 8}{\sqrt{14}}$$

$$\text{distance} = -\frac{28}{\sqrt{14}} \text{ or } -2\sqrt{14}$$

The result means that the actual distance from the given plane to the point is $2\sqrt{14}$ units to the scale of the drawing and that the point is on the same side of the given plane as the origin. If the given point and the origin are on opposite sides of the given plane, the result is positive because ρ is considered a positive quantity.

In Figure 14, P (x', y', z') is any given point on plane L'M'N' which is parallel to the given plane LMN. The line OD = ρ is the normal to the given plane LMN from the origin O. The resulting line OD intersects the plane L'M'N' in D'. OD' = ρ'. The line PK is the perpendicular distance from P to the given plane and the required distance from the given plane to the point P.

$$KP = DD' = OD' - OD = \rho' - \rho$$

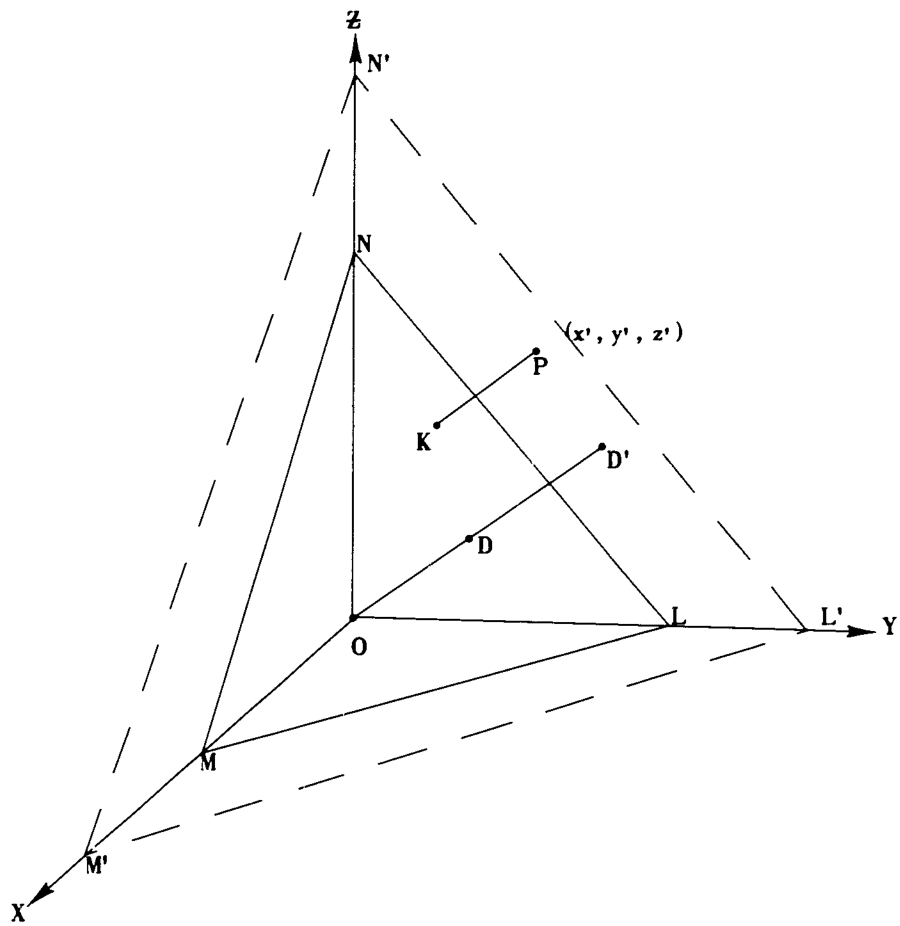

Figure 14

Since the second plane L'M'N' passes through the given point P, (x', y', z') its coordinates satisfy its equation,

giving

$$x' \cos\alpha + y' \cos\beta + \cos\gamma = \rho'.$$

Substituting this value of ρ' in $KP = \rho' - \rho$

$$KP = x' \cos\alpha + y' \cos\beta + z' \cos\gamma - \rho.$$

This equation is the same as the normal form of the equation of the given plane with the coordinates x' y' and z' substituted for x, y, and z.

4.4 Angles between Two Planes. The angle between two planes is the same as the angle between the directors of the two planes, Figure 15. The formula for finding the angle θ between the two planes whose general equations are

$A_x + B_y + C_z + D = 0$ and $A_1x + B_1y + C_1z + D = 0$ is

$$\cos\theta = \frac{AA_1 + BB_1 + CC_1}{\sqrt{A^2 + B^2 + C^2}\ \sqrt{A_1^2 + B_1^2 + C_1^2}}$$

To find the angle between the two planes $-x + 7y = 11$ and $3x + 4y + 5z = 10$ substitute in the formula

$$\cos\theta = \frac{(-1)(3) + (7)(4) + (0)(5)}{\sqrt{(-1)^2 + 7^2 + 0^2}\ \sqrt{3^2 + 4^2 + 5^2}} = \frac{25}{50} \text{ or } \tfrac{1}{2}$$

and the angle is 60°.

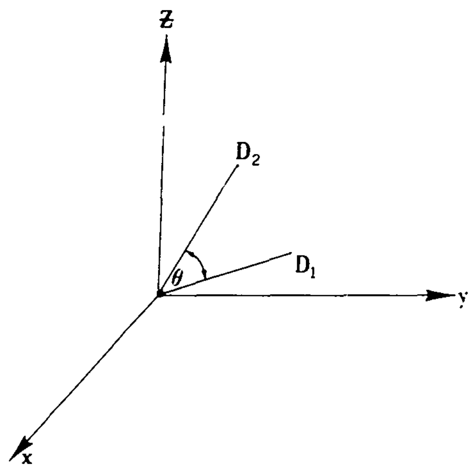

Figure 15

The angle between two planes is the same as the angle between the respective normals, OD_1 and OD_2. The planes are not shown in the drawing.

4.5 Parallel and Perpendicular Planes. Two planes whose equations are

$A x + B y + C z + D = 0$

$A'x + B'y + C'z + D' = 0$

are parallel when and only when the coefficients of x, y and z are proportional, that is $\frac{A}{A'} = \frac{B}{B'} = \frac{C}{C'}$.

For example the two planes

$$2x + 3y - z = 0$$

and $$4x + 6y - 2z + 8 = 0$$

are parallel because

$$\frac{2}{4} = \frac{3}{6} = \frac{1}{2}$$

Two planes are perpendicular when and only when the sum of the products of the coefficients of x, y, and z is zero, that is

$$AA' + BB' + CC' = 0$$

For example the two planes

$$2x + 3y - z = 0$$

and $$3x - y + 3z + 2 = 0$$

are perpendicular because

$$(2)(3) + (3)(-1) + (-1)(3) = 0$$

4.6 Special Planes. A plane whose equation has the form

$Ax + By + D = 0$ is perpendicular to the XY-plane

$By + Cz + D = 0$ is perpendicular to the YZ-plane

$Ax + Cz + D = 0$ is perpendicular to the ZX-plane.

That is, if one variable is lacking, the plane is perpendicular to the coordinate plane corresponding to the two remaining variables.

A plane whose equation has the form

$Ax + D = 0$ is perpendicular to the axis of x

$By + D = 0$ is perpendicular to the axis of y

$Cz + D = 0$ is perpendicular to the axis of z.

That is, with only one variable, the plane is perpendicular to that axis. For example, the plane $Ax + D = 0$ is perpendicular to both the XY-plane and ZX-plane, and hence is also perpendicular to the line of their intersection.

5. THE COORDINATE TRANSFORMATION IN SPACE AND SYSTEMS OF COORDINATES

5.1 Transformation of Rectangular Coordinates

5.1.1 Translation of the Axes. Given two parallel sets of coordinate axes, XYZ system and X'Y'Z'system, Figure 16, with the point P common to both coordinate systems. The origin O' of the X'Y'Z' system is the point (x_o, y_o, z_o) of the XYZ system, and the origin O of the XYZ system is the point $(-x_o, -y_o, -z_o)$ of the X'Y'Z'system. As in plane analytical geometry, the following equations are obtained:

$$
\begin{aligned}
x' &= x - x_o \qquad & x &= x' + x_o \\
y' &= y - y_o \qquad & y &= y' + y_o \\
z' &= z - z_o \qquad & z &= z' + z_o
\end{aligned}
\tag{5-1}
$$

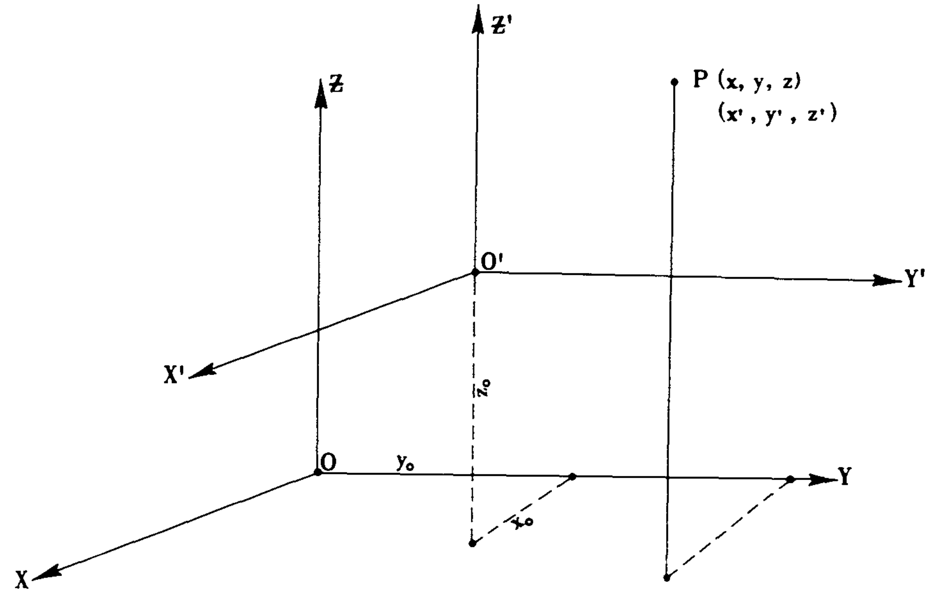

Figure 16

If the expressions for x, y, and z in the second set are substituted in an equation in x, y and z, the corresponding equation in x', y', z'is obtained. A transformation such as (5-1) is the translation of axes.

5.1.2 Rotation of Axes. Let the coordinates of a point P, referred to a set of rectangular axes OX, OY, OZ, be x, y, z, and referred to another system OX', OY', OZ' having the same origin, be x', y', z'. Let x' = OL', y' = L'M', z' = M'P (Figure 17); and let the direction cosines of Ox', referred to Ox, Oy, Oz, be λ_1, μ_1, ν_1; those of OY'be λ_2, μ_2, ν_2, and OZ' be λ_3, μ_3, ν_3, where

$$\lambda_i = \cos \alpha_i$$

$$\mu_i = \cos \beta_i$$

$$\nu_i = \cos \gamma_i \quad (i = 1, 2, 3).$$

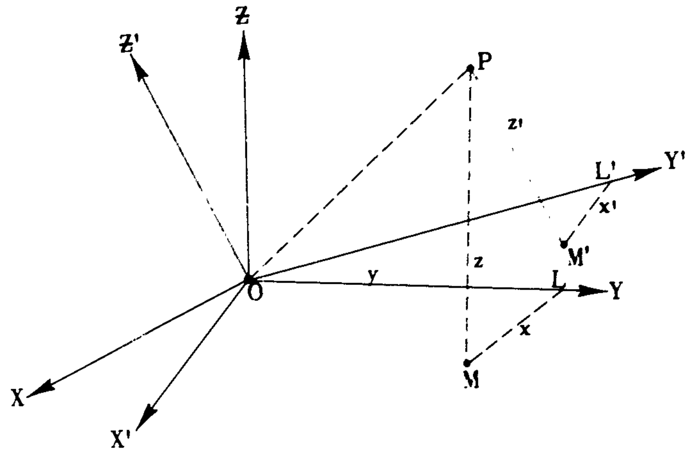

Figure 17

The projection of OP on the OX axis is x. The sum of the projections of OL', L'M'and M'P is $\lambda_1 x' + \lambda_2 y' + \lambda_3 z'$. This follows from the theorem that the length of the projection of a segment of a directed line on a second directed line is equal to the length of the given segment multiplied by the cosine of the angle between the lines. The second and third equations are obtained in a similar manner

$$\begin{aligned} x &= \lambda_1 x' + \lambda_2 y' + \lambda_3 z' \\ y &= \mu_1 x' + \mu_2 y' + \mu_3 z' \\ z &= \nu_1 x' + \nu_2 y' + \nu_3 z', \end{aligned} \tag{5-2}$$

where λ_i, μ_i, ν_i are the direction cosines of OX, OY, and OZ with respect to the OX', OY', and OZ' axes respectively. Project OP and OL = x, LM = y and MP = z on OX', OY', and OZ', then

$$\begin{aligned} x' &= \lambda_1 x + \mu_1 y + \nu_1 z \\ y' &= \lambda_2 x + \mu_2 y + \nu_2 z \\ z' &= \lambda_3 x + \mu_3 y + \nu_3 z \end{aligned} \tag{5-3}$$

The systems of equations (5-2) and (5-3) are expressed conveniently by the following diagram

	x'	y'	z'
x	λ_1	λ_2	λ_3
y	μ_1	μ_2	μ_3
z	ν_1	ν_2	ν_3

5.2 Systems of Coordinates.

5.2.1 Polar or Spherical Coordinate Systems.

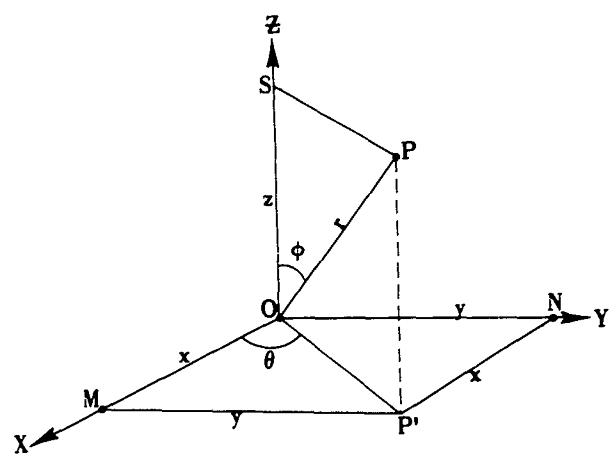

Figure 18

Let P' be an orthogonal projection of P on the XY-plane as shown in Figure 20. The position of P is defined by the distance r, the angle ϕ = ZOP which the line OP makes with the Z-axis, and the angle θ (measured by the angle XOP') which the plane through P and the Z-axis makes with the plane XOZ. The segment OM is the x distance along the OX-axis, ON the y distance along the OY-axis and OS the z distance along the OZ-axis. The numbers r, ϕ, θ are called the spherical coordinates of P; some writers call them polar coordinates in space. The length r is the radius vector, the angle ϕ is the co-latitude, and θ is called the longitude.

5.2.2 Transformation from Spherical Coordinates to Rectangular Coordinates and Vice Versa.

If P = (x, y, z), then from Figure 18,

$OP' = r \cos (90 - \phi) = r \sin \phi$

In triangle OMP', $x = OM = OP' \cos \theta = r \sin \phi \cos \theta$.

In triangle ONP', $y = ON = OP' \sin \theta = r \sin \phi \sin \theta$.

In triangle OP'P, $z = PP' = OP' \cos \phi = r \cos \phi$, or for rectantular coordinates

$$x = r \sin \phi \cos \theta$$
$$y = r \sin \phi \sin \theta$$
$$z = r \cos \phi$$

On solving for the spherical coordinates ϕ, θ, r in terms of the rectangular coordinates x, y, z

$$r = \sqrt{x^2 + y^2 + z^2}$$ (Distance between two points in space)

$$\phi = \arc\cos \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

$$\theta = \arc\tan x/y$$

5.2.3 Definition - Right-Handed-Rectangular Coordinate System.

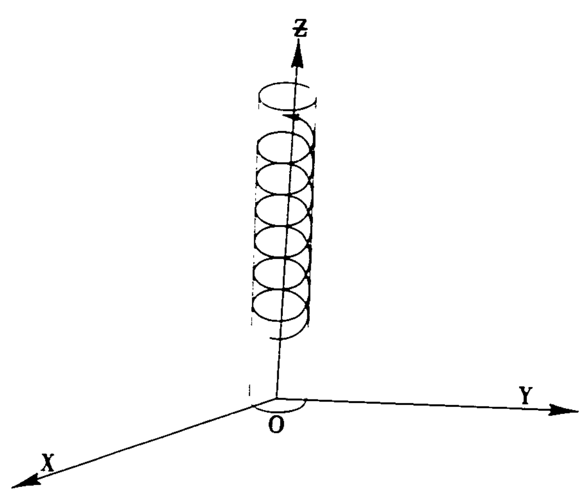

Figure 19

The right-handed rectangular coordinate system derives its name from the fact that a right threaded screw rotated through 90° from OX to OY will advance in the positive Z direction, as illustrated in Figure 19. The system is right-handed if and only if the rotation needed to turn the x-axis OX into the y-axis direction OY through an angle XOY $<$ 180 degrees would propel a right-handed screw toward the positive side of the of the XY-plane associated with the positive Z axis OZ.

5.2.4. Geocentric Coordinate System. A right-handed system with its origin at the center of the reference ellipsoid of the earth, the positive z axis in the direction of geographic north pole, the x and y axes in the plane of the equator, the positive x axis in the plane of the meridian through Greenwich, Figure 20.

5.2.5 Local Ground Coordinate System. A right-handed system with the x and y axes in a plane tangent to the reference ellipsoid of the earth, the positive y axis in the meridian plane through the origin and pointed to the North, the positive z axis pointed toward the zenith and containing the exposure station, Figure 20.

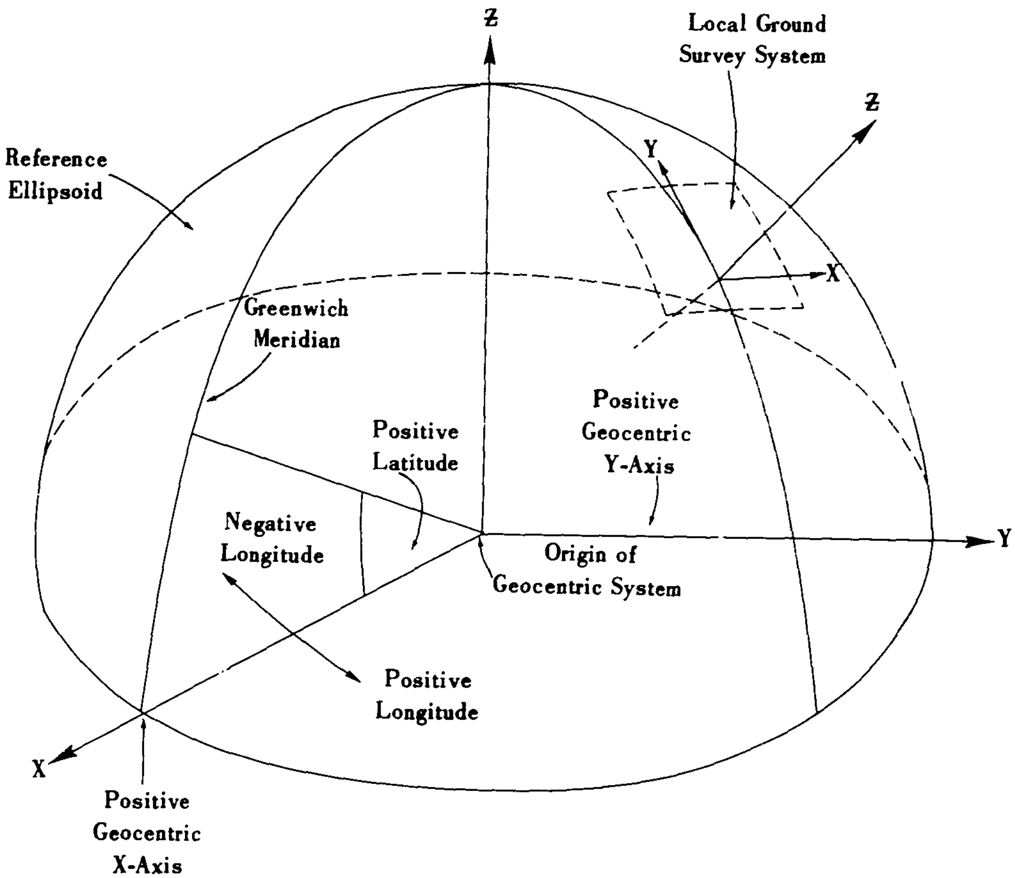

Figure 20

5.3 Use of Matrix Notation in Coordinate Transformation.

5.3.1 Definitions. A matrix is a rectangular array of numbers or functions (later defined as elements) of the form:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ a_{m_1} & a_{m_2} & \cdot & \cdots & a_{mn} \end{bmatrix}$$

where m represents the number of rows in the array and n the number of columns. This arrangement is defined here as matrix A. If m = n, the matrix is a square matrix. A matrix of which all elements except the diagonal elements are equal to zero is called a diagonal matrix:

$$\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & a_{mm} \end{bmatrix}$$

A unit matrix is a diagonal matrix whose diagonal elements are all 1's, and a null matrix is one in which all of the elements are equal to zero.

5.3.2 Addition and Subtraction. The sum or difference of two matrices is a new matrix formed by adding or subtracting, the elements which lie in corresponding positions, such as

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ a_{m_1} & \cdots & a_{mn} \end{bmatrix} \pm \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ b_{m_1} & \cdots & b_{mn} \end{bmatrix} = \begin{bmatrix} a_{11} \pm b_{11} & \cdots & a_{1n} \pm b_{1n} \\ \cdot \quad \cdot & \cdots & \cdot \quad \cdot \\ \cdot \quad \cdot & \cdots & \cdot \quad \cdot \\ a_{m_1} \pm b_{m_1} & \cdots & a_{mn} \pm b_{mn} \end{bmatrix}$$

or

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ a_{m_1} & \cdots & a_{mn} \end{bmatrix} - \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ b_{m_1} & \cdots & b_{mn} \end{bmatrix} = \begin{bmatrix} a_{11} - b_{11} & \cdots & a_{1n} - b_{1n} \\ \cdot \quad \cdot & \cdots & \cdot \quad \cdot \\ \cdot \quad \cdot & \cdots & \cdot \quad \cdot \\ a_{m_1} - b_{m_1} & \cdots & a_{mn} - b_{mn} \end{bmatrix}$$

By using the letters A, B, and C respectively to represent the above matrices, these operations can be written as

$$A + B = C \text{ or } A - B = C$$

5.3.3 <u>Multiplication of Matrices</u>. The product of two matrices is defined as the new matrix formed by multiplying the corresponding row elements of the first matrix by the corresponding column elements of the second matrix and adding the results:

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \quad x \quad \begin{bmatrix} b_{11} & \cdots & b_{1m} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ b_{n1} & \cdots & b_{nm} \end{bmatrix}$$

$$\begin{bmatrix} a_{11}\, b_{11} + \cdots + a_{1n}\, b_{n1} & \cdots & a_{11}\, b_{1m} + \cdots a_{1n}\, b_{nm} \\ \cdot \quad \cdot \quad \cdot \cdots \cdot \quad \cdot \quad \cdot & \cdots & \cdot \quad \cdot \quad \cdot \cdots \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \cdots \cdot \quad \cdot \quad \cdot & \cdots & \cdot \quad \cdot \quad \cdot \cdots \cdot \quad \cdot \\ a_{m1}\, b_{11} + \cdots \quad a_{mn}\, b_{n1} & \cdots & a_{m1}\, b_{1} + \cdots a_{mn}\, b_{nm} \end{bmatrix}$$

or written in shorthand form

$$A \times B = C$$

In order that matrix multiplication of A times B be possible, the matrices A and B must be conformable, that is, the number of rows in A must equal the number of columns in B. If A is of the size m by p, that is m rows and p columns, and B is p by n, then C is of size m by n. To multiply a matrix A by a number k, each element in A must be multiplied by k, or

$$k \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11}k & a_{12}k & a_{13}k \\ a_{21}k & a_{22}k & a_{23}k \\ a_{31}k & a_{32}k & a_{33}k \end{bmatrix}$$

5.3.4 Examples of Matrix Notation in Linear Transformation Equations. Matrix notations are commonly used to represent linear transformation equations, involving translation and rotation of coordinate axes. The following is a set of such equations with matrix notation used in geodetic coordinate transformation. Numbers in parenthesis indicate the size (mn) required for the matrices to be conformable:

(1)
$$x = x' \cos\alpha - y \sin\alpha$$
$$y = x' \sin\alpha + y \cos\alpha$$
$$z = z'$$

Matrix Notations:

$$\underset{(3x1)}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \underset{(3x3)}{\begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{(3x1)}{\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}} \quad \text{or}$$

by letter representation for each matrix

$$X = AX'$$

(2)
$$P = P_o + y' \cos\phi + z' \sin\phi$$
$$Q = Q_o - y' \sin\phi + z' \cos\phi$$
$$R = R_o + x'$$

Matrix Notation:

$$\underset{(3x1)}{\begin{bmatrix} P \\ Q \\ R \end{bmatrix}} = \underset{(3x1)}{\begin{bmatrix} P_o \\ Q_o \\ R_o \end{bmatrix}} \underset{(3x3)}{\begin{bmatrix} 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \\ 1 & 0 & 0 \end{bmatrix}} \underset{(3x1)}{\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}} \quad \text{or}$$

by letter representation for each matrix

$$P = P_o + AX'$$

(3)
$$x' = (R - R_o)$$
$$y' = (P - P_o) \cos\phi - (Q - Q_o) \sin\phi$$
$$z' = (P - P_o) \sin\phi + (Q - Q_o) \cos\phi$$

Matrix Notation:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} P - P_o \\ Q - Q_o \\ R - R_o \end{bmatrix} \quad \text{or}$$

$$\underset{(3x1)}{} \qquad \underset{(3x3)}{} \qquad \underset{(3x1)}{}$$

by letter representation for each matrix

$$x' = AP - AP_o$$

(4)
$$x = x' + Ak(x - x_p) + A'k(y - y_p) + D'kc$$
$$y = y' + Bk(x - x_p) + B'k(y - y_p) + E'kc$$
$$z = z' + ck(x - x_p) + c'k(y - y_p) + F'kc$$

Matrix Notation:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} + k \begin{bmatrix} A & A' & D' \\ B & B' & E' \\ C & C' & F' \end{bmatrix} \begin{bmatrix} \bar{x} - x_p \\ \bar{y} - y_p \\ C \end{bmatrix} \quad \text{or}$$

$$\underset{(3x1)}{} \qquad \underset{(3x1)}{} \qquad \underset{(3x3)}{} \qquad \underset{(3x1)}{}$$

by letter representation for each matrix

$$x = x' + k$$

(5) Direction Cosines relative to XYZ System

$$X = \lambda_1 x' + \lambda_2 y' + \lambda_3 z'$$
$$Y = \mu_1 x' + \mu_2 y' + \mu_3 z'$$
$$Z = \nu_1 x' + \nu_2 y' + \nu_3 z'$$

Matrix Notation:

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \\ \nu_1 & \nu_2 & \nu_3 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$

(3x1) (3x3) (3x1) or by

letter representation for each matrix

$$X = \lambda x'.$$

6. PRACTICAL APPLICATIONS

6.1 Introductory Statement. Many problems of analytical triangulation can be solved by a series of coordinate transformations. Vector and matrix algebra have proved to be most useful for this purpose when the work is to be done by electronic computers. This section provides a useful introduction to the subject of coordinate transformation.

6.2 The Aerial Photograph. An aerial photograph is always considered to be tilted. The observed rectangular coordinates of the image on a tilted photograph are called x, y and z with z as the focal length of the aerial camera. The photograph should be considered a diapositive with the x- and y- axes perpendicular lines defined on the photograph by the images of the fiducial marks in the camera. The positive x- axis is the axis which most closely coincides with the direction of the flight line. The y- and z- axes are chosen to make a right-handed coordinate system. Thus the image-space x, y, z coordinnates of the nodal point are 0, 0, –f where f is the focal length of the lens.

In certain instances where the measurements are to be made on the emulsion side of the original negative (as with glass plate negatives) the positive direction of the y-axis should be reversed. The image space x, y, z coordinates of the nodal point for this case are 0, 0, f.

6.3 Three Pairs of Conjugate Axes. In addition to the x,y,z coordinate system which identifies the position of the image on an aerial photograph, an auxiliary system (x',y',z') is introduced. The x',y',z' system has the same origin as the x,y,z system and differs from it only in orientation. In this regard, the x',y',z' axes are selected parallel to the X,Y,Z axes of the local (geodetic or geocentric) ground coordinate system where + Z is upward and X and Y form a right-handed system such that +X is East and +Y is North. The angular orientation of one system with the other is completely specified by the angles between three pairs of conjugate axes.

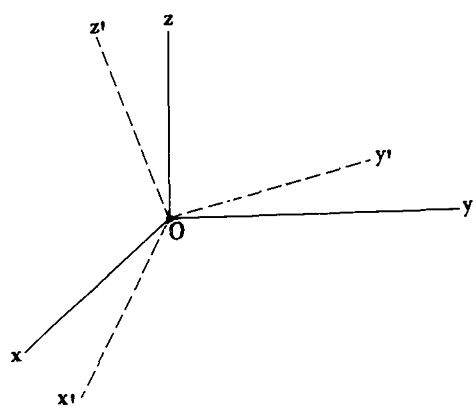

Figure 21

Two independent systems with a common origin

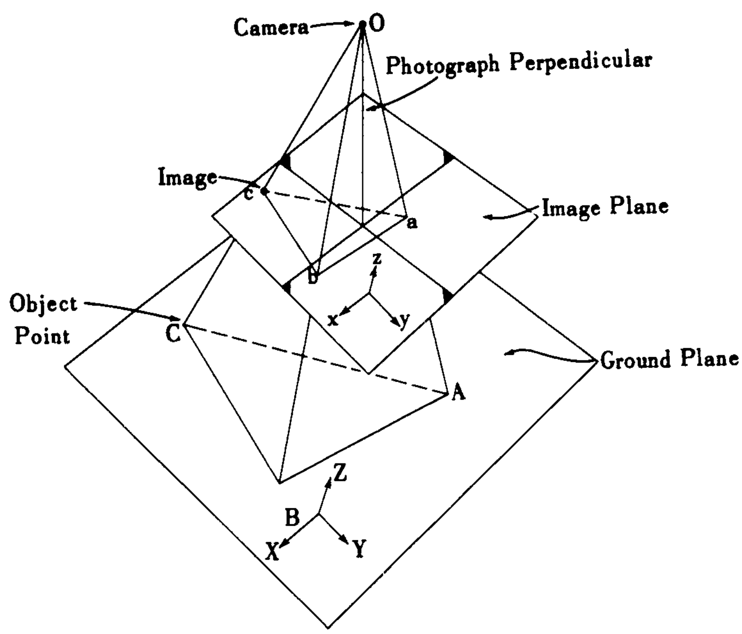

Figure 22

Figure 22 shows a tilted aerial photograph with the images a, b, and c of three ground objects A, B, and C. The ground objects can be regarded as control stations for which the X- and Y- coordinates are known with respect to some oriented geographic system, and the elevation Z is also known relative to a recognized horizontal plane such as sea level. The geographic coordinates of the object A may be symbolized as X_A, Y_A, and Z_A, and the same for the B and C. The perspective center O of the photograph is also considered to have geographic coordinates X_o, Y_o, Z_o in the same ground system. It is convenient in most problems to regard O as the origin of the system of ground coordinates.

Figure 23 illustrates the geometric relationship between a spatial object A, a photographic image B as recorded on an image plane, and a camera station C. In the illustration the camera station (C) is the center of projection of a three dimensional space object (A) on a two dimensional image plane (B) and the points A, B and C are considered to be on a straight line.

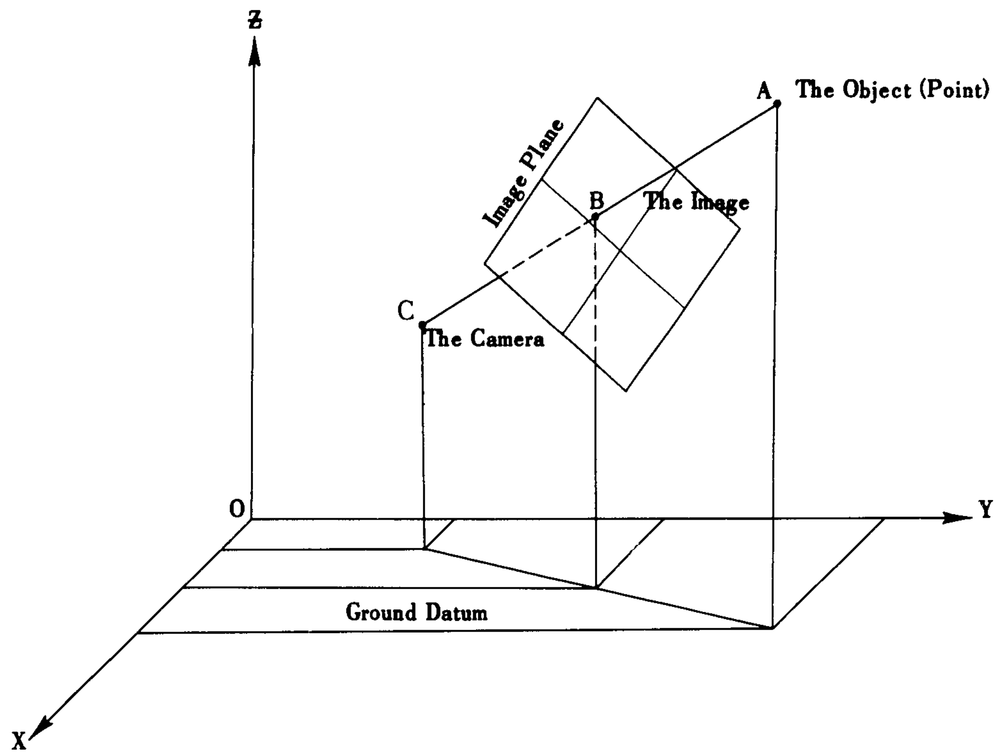

Figure 23

6.4 The Three (Euler) Rotation Angles ω, ϕ and K. The angles ω, ϕ and K are defined by three successive rotations of the initial set of coordinates x,y,z into an auxiliary set of coordinates x',y',z'. The positive direction of the rotations is considered counterclockwise about the positive direction of the area.

The first rotation, omega (ω), or x- tilt is about the x-axis (primary axis) considered counterclockwise positive as viewed from the plus end of the x- axis (Rosenfield considers the rotation clockwise as viewed from the origin which is physically identical). Omega (ω) is the dihedral angle between the photo perpendicular and the XZ plane. That component of tilt which is measured around the x- axis is called roll. Positive roll occurs when the right wing of the aircraft is lowered in flight.

The second rotation (Figure 24), phi (ϕ), or y- tilt is about the inclined y- axis (secondary axis). Phi (ϕ) is the dihedral angle between the photo perpendicular and the ZY plane. That component of tilt which is measured around the y- axes is called pitch. Positive pitch occurs when the nose of the aircraft is lowered during flight.

The third rotation, Kappa (K), or swing is the rotation about the twice rotated z' axis (tertiary axis) which is the perpendicular lens axis or the z- axis of the photograph.

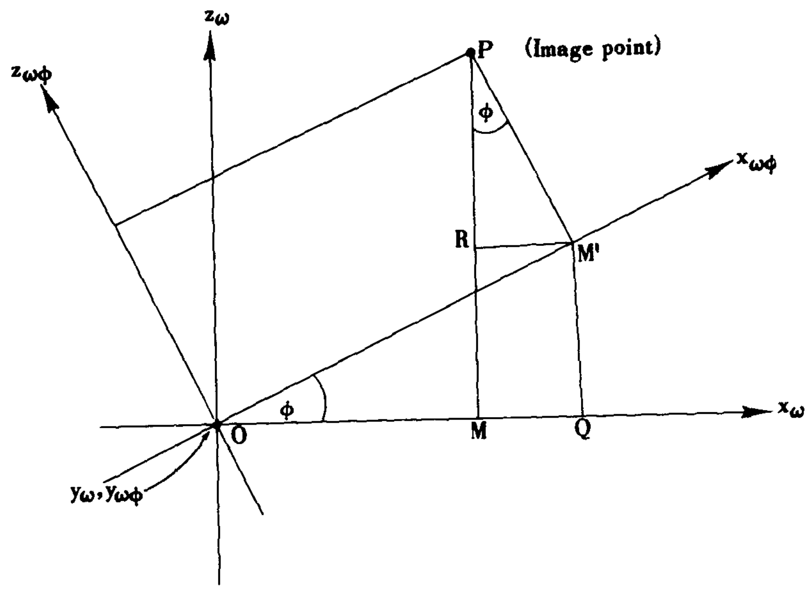

Figure 24 The ϕ - rotation

$$x_\omega = OM = OQ - MQ$$
$$= OQ - RM'$$
$$z_\omega = MP = MR + RP$$
$$= QM' + RP$$
$$OQ = OM' \cos\phi = x_{\omega\phi} \cos\phi$$
$$QM' = OM' \sin\phi = x_{\omega\phi} \sin\phi$$
$$RP = M'P \cos\phi = z_{\omega\phi} \cos\phi$$
$$RM' = M'P \sin\phi = Z_{\omega\phi} \sin\phi$$
$$x_\omega = x_{\omega\phi} \cos\phi - z_{\omega\phi} \sin\phi$$
$$y_\omega = y_{\omega\phi}$$
$$z_\omega = x_{\omega\phi} \sin\phi + z_{\omega\phi} \cos\phi$$

Put in matrix form, the last three equations are:

$$\begin{bmatrix} x_\omega \\ y_\omega \\ z_\omega \end{bmatrix} = \begin{bmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{bmatrix} \begin{bmatrix} x_{\omega\phi} \\ y_{\omega\phi} \\ z_{\omega\phi} \end{bmatrix}$$

The matrix form for the omega and kappa rotations are derived in the same way. These angles can most conveniently be obtained in terms of the three direction cosines of each of the three axes of one system in terms of the other. The nine cosines are sometimes spoken of as the vectorial elements of space orientation.

6.5 Matrix Form of Rotation Formulae--Direction Cosines. The rotation formula can readily be expressed in matrix form. (See Figure 24.)

The matrix for the primary rotation through the angle omega (ω) about the x-axis is

$$A(\omega) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\omega & \sin\omega \\ 0 & -\sin\omega & \cos\omega \end{bmatrix}$$

This rotation generates a set of coordinates which is then transformed into another set by the secondary rotation through the angle phi (ϕ) about the y- axis.

The matrix for the secondary rotation about the new position of the y- axis is

$$A(\phi) = \begin{bmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{bmatrix}$$

This rotation generates another set of coordinates which is then transformed into still another set by the tertiary rotation through the angle kappa (K) about the z-axis. The matrix for the tertiary rotation about the new position of the z- axis is

$$A(K) = \begin{bmatrix} \cos K & \sin K & 0 \\ -\sin K & \cos K & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The three successive rotations must, of course, correspond to a single orthogonal transformation. Since the sub-matrices (A_ω, A_ϕ, AK) refer to a counter-clockwise rotation of a coordinate system the sequential matrix multiplication to form a single matrix must be clockwise. (AK, A_ϕ, A_ω) as follows:

$$\begin{bmatrix} \cos K & \sin K & 0 \\ -\sin K & \cos K & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\omega & \sin\omega \\ 0 & -\sin\omega & \cos\omega \end{bmatrix}$$

Applying the rule of matrix multiplication the following nine elements or direction cosines are obtained:

$$\begin{bmatrix} \cos\phi \cos K & \cos\omega \sin K + \sin\omega \sin\phi \cos K & \sin\omega \sin K - \cos\omega \sin\phi \cos K \\ -\cos\phi \sin K & \cos\omega \cos K - \sin\omega \sin\phi \sin K & \sin\omega \cos K + \cos\omega \sin\phi \sin K \\ \sin\phi & -\sin\omega \cos\phi & \cos\omega \cos\phi \end{bmatrix}$$

6.6 Equations Relating the Two Systems (x, y, z and x', y', z') of Rectangular Coordinates and the Nine Direction Cosines. The relations between the nine direction cosines and two sets of rectangular spatial coordinates (x, y, z and x', y', z') can be tabulated conveniently in the following conventional manner:

$$\begin{array}{c|ccc} & x' & y' & z' \\ \hline x & a_{11} & a_{12} & a_{13} \\ y & a_{21} & a_{22} & a_{23} \\ z & a_{31} & a_{32} & a_{33} \end{array} \qquad (6.1)$$

Exactly the same information is given by conventional vector notation:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} \qquad (6.2)$$

The inverse notation is also useful:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Many forms are used in mathematical literature to represent an algebraic matrix. In this text a_{mn} represents a matrix having mn elements, arranged in m rows and n columns. As explained in section 5.3 the first subscript indicates the row in which each element is located while the second subscript indicates the column in which it is located. According to this notation, the element a_{23} will be placed in the second row of the third column of its matrix. The range of the subscripts determining the number of rows and columns defines the type or order of a matrix.

The elements in Equation 6.2 expressed in terms of the direction cosines as given on the top of page 40 are as follows:

$$a_{11} = \cos\phi \cos K$$

$$a_{12} = \cos\omega \sin K + \sin\omega \sin\phi \cos K$$

$$a_{13} = \sin\omega \sin K - \cos\omega \sin\phi \cos K$$

$$a_{21} = -\cos\phi \sin K$$

$$a_{22} = \cos\omega \cos K - \sin\omega \sin\phi \sin K$$

$$a_{23} = \sin\omega \cos K + \cos\omega \sin\phi \sin K$$

$$a_{31} = \sin\phi$$

$$a_{32} = -\sin\omega \cos\phi$$

$$a_{33} = \cos\omega \cos\phi$$

The Perspective Transformation

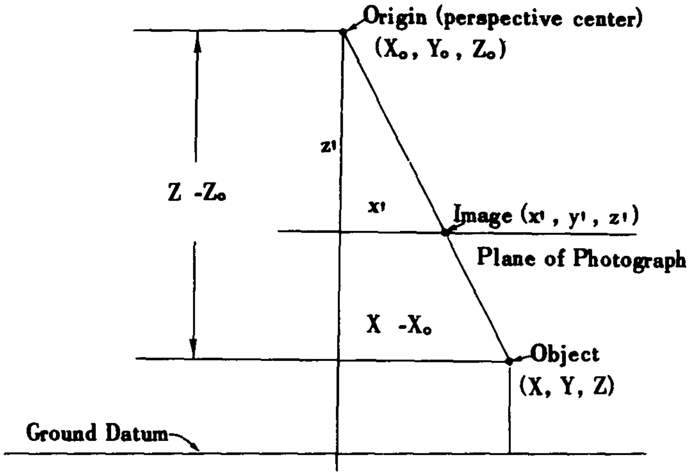

Figure 25

In this x' y' z'- coordinate system the image plane is considered parallel to the XY-plane. It is evident from the similar triangles in the figure that

$$(X - X_o)/(Z - Z_o) = x'/z' \text{ from which } x' = (X - X_o)z'/(Z - Z_o)$$

Similarly, by visualizing the YZ-plane,

$$(Y - Y_o)/(Z - Z_o) = y'/z' \text{ from which } y' = (Y - Y_o)z'/(Z - Z_o)$$

By algebraic identity

$$(Z - Z_o)/(Z - Z_o) = z'/z' \text{ from which } z' = (Z - Z_o)z'/(Z - Z_o)$$

By substituting these values for x', y' and z' in Equation 6.2 and dividing the expression for x and y by that for z we obtain the following basic projective transformation equations:

$$\frac{x}{z} = \frac{(X - X_o)a_{11} + (Y - Y_o)a_{12} + (Z - Z_o)a_{13}}{(X - X_o)a_{31} + (Y - Y_o)a_{32} + (Z - Z_o)a_{33}}$$

$$\frac{y}{z} = \frac{(X - X_o)a_{21} + (Y - Y_o)a_{22} + (Z - Z_o)a_{23}}{(X - X_o)a_{31} + (Y - Y_o)a_{32} + (Z - Z_o)a_{33}} \qquad (6.3)$$

where X,Y,Z are the coordinates of an object on the ground, X_o, Y_o,Z_o are the coordinates of the camera station in the same system and x,y,z are the image-coordinates in which $z = -f$ the camera focal length. Equation 6.3 expresses the condition that the photographic image, the ground object, and the camera lens point are colinear. Equation 6.3 is transcendental and, in the most general case, all twelve terms are considered as unknowns. Consequently a form of Newton's method is used to solve them. This is an iterative method based on initial approximations which are quite easily obtained for all the unknowns. The first approximation of Phi (ϕ) and Omega (ω) is always zero. Kappa (K) can be measured with a protractor on a stapled mosaic or a photo-index. The flying height Z is known. The X and Y can be measured with sufficient accuracy for the problem from a photo-index. After these substitutions are made, the computation makes all the other corrections by successive approximations. Usually three iterations are required.

The above equations are too involved to be solved for each of the six parameters in the direct manner and a subroutine computer program is required.

In Equation (6.1) a_{11}, a_{12} and a_{13} denote the direction cosines between the image-space x- axis and the object-space X Y Z axis respectively. The three direction cosines of the y- axis in the X Y Z system are a_{21}, a_{22} and a_{23} and similarly for the z-axis. The three direction cosines of the X- axis in the x y z system of coordinates are also a_{11}, a_{21} and a_{31} and similarly for the other axis.

a_{11} is the cosine of the angle between the X- and x- axes, a_{22} is the cosine of the angle between the Y- and y- axes, and a_{33} is the cosine of the angle between the Z- and z- axes. Incidentally a_{33} is the cosine of the tilt angle because the z- axis is the photo perpendicular and the Z- axis is the plumb line.

The nine direction cosines of the matrix are not totally independent values. The sum of the squares of the elements of any row or any column of the matrix equals one. The sum of the products of corresponding elements of any two columns or any two rows is zero, or more specifically,

$$a_{11}\,a_{21} + a_{12}\,a_{22} + a_{13}\,a_{23} = 0.$$

The value of the determinant of the matrix is either + 1 or − 1.

The elements of the matrix are the elements of a linear orthogonal transformation because they enable the computation of the coordinates of a point in either system from the given coordinates of the same point in the other system. If x, y and z are known for a point, and if the nine elements of the matrix are known then

$$X = xa_{11} + ya_{21} + za_{31}$$

or if X, Y and Z are known for a point, and if the nine elements of the matrix are known then

$$X = Xa_{11} + Ya_{12} + Za_{13}$$

6.7 Inverse Case -- Object Space Coordinate System in Terms of the Image Space Coordinate System. The matrix expression in Equation 6.2 has been developed for the condition wherein the image-space coordinate system is expressed in terms of the object-space coordinate system. The coordinates x, y and z can always be measured in the image coordinate system. If the inverse case is desired, the object-space coordinate system in terms of the image-space coordinate system, the final orientation matrix must be inverted. The matrix can be inverted directly by transposition of its elements since, for an orthogonal matrix, the inverse is equal to its transpose. The inverse form can also be developed by inversion of each of the individual rotational submatrices which would necessitate inverting the order of matrix multiplication.

6.8 Relation Between the Photographic Coordinate System (Exposure Station) and the Geocentric Coordinate System (Local Survey). The following Equations relate the two coordinate systems:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \qquad (6.4)$$

where x y z are photo coordinates X Y Z are geocentric coor 'inates

λ = longitude (positive in value east from Greenwich)

ϕ = latitude (positive in value north of the equator)

and where $a_{11} = \sin\lambda$

$a_{12} = \cos\lambda$

$a_{13} = 0$

$a_{21} = -\sin\phi \cos\lambda$

$a_{22} = \sin\phi \sin\lambda$

$a_{23} = \cos\phi$

$a_{31} = \cos\phi \cos\lambda$

$a_{32} = -\cos\phi \sin\lambda$

$a_{33} = \sin\phi$

In the geocentric coordinate system, the Z axis is toward the north pole, the +X and +Y axis lie in the plane of the equator with the +X axis in the meridian plane through Greenwich. Therefore, the cosine of λ is minus for longitudes of +90° to +180° and latitude of 0° to +90°. Consider λ negative in the western hemisphere. Formulas for this transformation are derived by rotating and translating modified geocentric coordinates. The classical geocentric coordinates are:

$$X = (N + h) \cos \phi \sin \lambda$$

$$Y = (N + h) \cos \phi \cos \lambda$$

$$Z = [N(1 - e^2) + h] \sin \phi$$

where ϕ and λ are the latitude and longitude of any point, h is the elevation, and N is the length of the normal through ϕ, λ.

Resection in photogrammetry is defined as the determination of the six fundamental parameters ω, ϕ, K, X_o, Y_o, Z_o of a single photograph from the given positions and elevations of at least three non-colinear points imaged on the photograph. The X, Y, Z-coordinates in Equation (6.4) are in the form required for resection. The final adjusted X, Y, Z's need to be transformed back to latitudes, longitudes and elevations. For this the following formulae are needed:

$$\tan \lambda = X/Y$$

$$\tan \phi = Z/(X^2 + Y^2)^{1/2}$$

$$Z = (N + h) \sin \phi$$

$$h = (X/\cos \phi \sin \lambda) - N \quad \text{or}$$

$$h = (Y/\cos \phi \cos \lambda) - N$$ (use the equation with the larger function involving λ)

6.9 Orientation. Since orientation is most important in this analytic system, the following brief review of the orientation phase is included.

The orientation phase can be divided into interior orientation and exterior orientation. Exterior orientation can be further subdivided into relative orientation and absolute orientation.

Interior orientation imposes two requirements:
(1) the principal distance of the diapositive must equal the principal distance of the projector, which is the length of the perpendicular from the interior perspective center of the projection lens to the plane of the diapositive; and
(2) this perpendicular must intersect the plane of the diapositive at the principal point of the photograph. This is the operation known as centering the diapositive.

Relative orientation is defined as the determination of the three angular and two linear parameters that specify the attitude and position of the photograph (camera station) with respect to another (overlapping) one that shows a sufficiently large common area.

Relative orientation is perhaps the most important item in this analytic system: it embodies all the basic mathematics that is peculiar to the system, is utilized again later in resection and the block adjustment, and requires the second-largest computer effort. It is in relative orientation that the principles of projective geometry are applied. A classic geometric rotation of the axes in three dimensions is needed in relative orientation to express the attitude of one photograph to another. Instead of using the three angles between the respective axes as in analytic geometry, a system of three sequential rotations is used, the primary one ω about a horizontal x-axis, the secondary one ϕ about the once rotated y-axis, and the tertiary one K about the camera axis.

Absolute orientation is the orientation with reference to a vertical and a horizontal datum. Absolute orientation procedure comprises three operations: (1) bringing the model to the proper size, or scaling the model; (2) leveling or horizontalizing the model; and (3) positioning the model with reference to the horizontal datum.

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